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Special Finsler Spaces – A Review of Literature

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Abstract-- The history of development of Finsler Geometry began in the year 1924 when three geometricians, viz. J. H. Taylor, J. L. Synge and L. Berwald worked in this field. The term "Finsler space" evokes in most mathematicians the picture of an impenetrable forest whose entire vegetation consists of tensors. Finsler spaces were discovered by Riemann in his lecture: *Über die Hypothesen, welche der Geometrie zu Grunde liegen* (1854). The goal which Riemann set for himself was the *definition and discussion of the most general finite-dimensional space in which every curve has a length derived from an infinitesimal length or line element.* Later on, Berwald gave the concept of Finsler space. The present work deals with the review of the works carried out in the field of Finsler space, with emphasis on special Finsler Spaces.

Keywords: Finsler Geometry, Finsler Space.

1. Introduction:

The term "Finsler space " evokes in most mathematicians the picture of an impenetrable forest whose entire vegetation consists of tensors. It shows that *the association of tensors* (or differential forms) *with Finsler spaces is due to an historical accident, and that,* at least at the present time, *the fruitful and relevant problems lie in a different direction.* Finsler spaces were discovered by Riemann in his lecture **[**22**]:** *Über die Hypothesen, welche der Geometrie zu Grunde liegen* (1854). The goal which Riemann set for himself was the *definition and discussion of the most general finite-dimensional space in which every curve has a length derived from an infinitesimal length or line element.*

Finsler spaces can be thought of as generalizations of Riemmanian manifolds; tangent spaces carry Minkowski norms instead of inner products and geometric objects on tangent vectors depend not only on the base but also on the fibre component. Chern and Shen have provided an authoritative treatment of the subject, for which treatise the book by Bao, Chern and Shen [19] is a helpful introduction with a wealth of detail. Finsler spaces have intrinsic geometrical significance and also they have been used to model a variety of problems from dynamics, optics, ecology and relativity, e.g. [20] and [19]. In 1986, [17] published his book which dealt with special Finsler Spaces, e.g. Berwald space, Minkowskian space, C-Reducble Finsler space, Randers space, etc. The present work deals with the review of the works being carried out in the field of special Finsler spaces.

2. Literature Review

[1] defined the generalized BR recurrent space. The purpose of the present paper to obtain the necessary and sufficient condition for (i) Berwald curvature tensor H^i_{jkh}, its associative H^j_{pkh} and Cartan's fourth curvature tensor to be generalized recurrent, (ii) the tensor (H_{hk} - H_{kh}) and H-Ricci tensor H_{kh} are to be non-vanishing and (iii) the torsion tensor K^i_{jk}, the deviation tensor K^i_h, K-Ricci tensor K_{jk}, the curvature vectors K_k, R_j and the curvature scalar H to behave as recurrent. Also to study the covariant vectors λ_m and μ_m.

[2] deals with study of recurrence of generalized second order in a Finsler space F^*_n equipped with non-symmetric connection. We have tried to make study of special birecurrent and special generalized birecurrent F^*_n of first and second kinds and have obtained results of significance. An attempt has also been made to study the recurrence of third order in such a Finsler space with special reference to the first curvature tensor.

[3] discussed that in the year 1979, M. Matsumoto has discussed non-Riemannian Finsler spaces with vanishing T-tensor where it was shown that if a Finsler space satisfy the T-condition i.e. , then for such a Finsler space the function of is a function of position only (i.e.), where L is the fundamental function and is square of length of torsion tensor . In continuity of the above it was further studied that Finsler spaces, is a function of x in detail. They further considered combination of L and C differently and taking $L^4{}_C = \gamma^3$, where γ is a well known cubic metric.

[4] introduced a Finsler space whose Cartan's fourth curvature tensor satisfies the condition $B_m K^i_{jkh} = \lambda_m K^i_{jkh} + \mu_m(\delta^i_h g_{jk} - \delta^i_k g_{jh}),\ K^i_{jkh} \neq 0,$ where λ_m and μ_m are non-zero covariant vectors field called *recurrence vector*. The space satisfying this condition will be called *a generalized BK-recurrent space*.

The purpose of this paper is to obtain Berwald covariant derivative of first order for torsion tensor and the deviation tensor , also to show that K- Ricci tensor , the curvature vectors , and the curvature scalar *H* are non-vanishing in our space.

[5] In this paper some results on the third order recurrent Finsler spaces have been obtained.

[6] introduced and studied a Finsler space, called *special C^v-reducible*, for which the Cartan *v*-covariant derivative of the Cartan tensor C_{ijk} is written in a special form. A necessary and sufficient condition for a special semi-*C*-reducible Finsler space to be special C^v-reducible is determined. Finally, a three dimensional special C^v-reducible Finsler space illustrates the developed theory.

[7] considered special form of v(hv) -torsion tensor, called such a Finsler space as *P*- Reducible Finsler space. Also we have worked out the role of *P*- Reducible Finsler spaces in other special Finsler spaces. They found out that the special Finsler Spaces can be applied in various branches of theoretical & computational Physics, theory of anisotropic media, lagrangian mechanics, to solve optimization problems, theory of Ecology, Theory of evolution of Biological Systems, in describing the internal symmetry of Hedrons, theory of Space Time & Gravitation, deformation of crystalline media, Seismic Phenomena, interfaces in thermodynamics system etc.

[8] developed the conformal curvature tensor by studying their properties in generalized BR-recurrent space. In generalized BR-recurrent Finsler space a R-Ricci tensor R_{jk} and the curvature vector R_k behave as recurrent.

[9] studied the class of of *C*3-like Finsler metrics which contains the class of semi-C-reducible Finsler metric. We find a condition on *C*3-like metrics under which the notions of Landsberg curvature and mean Landsberg curvature are equivalent.

[10] studied generalized symmetric Berwald spaces. We show that if a Berwald space (M, F) admits a parallel s−structure then it is locally symmetric. For a complete Berwald space which admits a parallel s-structure we show that if the flag curvature of (M, F) is everywhere nonzero, then F is Riemannian.

[11] here the concept of C^h-recurrent Finsler space have been studied by Makoto Matsumoto. The purpose of present paper is to study the properties of C^h recurrent (C^v-recurrent) torsion tensor field of second order in the Finsler spaces.

[12] studied the properties of hypersurfaces immersed in C2-like Finsler spaces. We prove that each non-Riemannian hypersurface of a C2-like Finsler space is C2-like. The condition under which a hypersurface of a C2-like Landsberg space is Landsberg is obtained. Finally after using the so called T-conditions we explore the situation in which a hypersurface of a C2-like Finsler space F_n satisfying the T-conditions also satisfies the T-condition.

[13] introduced the concept of (α, β)-metric and a number of propositions and theorems have worked for a (α, β)-metric.

[14] studied hypersurfaces of special Finsler spaces and also to investigate the various kinds of hypersurfaces of Finsler space with special (α,β) metric.

[15] dealt with the short review is concerned with real finite-dimensional Finsler manifolds (M, F) with Finsler structures F : TM [0, infinity) that satisfy the Landsberg conditions. The aim is to provide an annotated collection of references to geometric results that seem important in the study of Landsberg spaces and to suggest some areas for further work in this context.

[16] discussed the theory of conformal change of some special Finsler spaces namely C-reducible, semi C-reducible and C2-like five dimensional Finsler spaces with constant unified main scalar. We have obtained the values of main scalars and v-scalar curvature S of all the above spaces with constant unified main scalar.

[21] introduced and studied the concept of (λ,β), metric and a number of propositions and theorems have been workout for a (λ,β), metric, where λ^m ias a m^{th} root metric and β is a one form metric. They examined some conditions that characterize the normalized supporting element l^i , angular metric tensor h ji , metric tensor g ij and (h) hvtorsion tensor C ijk , The reciprocal of the metric tensor g_{ij} ,. T-tensor T_{hijk} , C-reducile of Finsler space with ((λ,β)-metric.

[23] In the present paper, we extend the notion of bienergy functional to curves on Finsler spaces and deduce the equations of biharmonic curves as the EulerLagrange equations attached to this functional; these equations turn out to involve, besides the flag curvature (which is the analogue of the curvature term in the Riemannian biharmonic equations), two specific, non-Riemannian quantities, namely, the Cartan tensor and the Landsberg tensor of the space. Still, just as in Riemannian spaces, all geodesics are biharmonic curves, but the converse is generally not true. A brief analysis in the Frenet frame shows that, though the equation of biharmonic curves becomes more complicated compared to its Riemannian counterpart, a basic result in Riemannian biharmonicity theory remains true in our case, namely, any Finslerian biharmonic curve has constant geodesic curvature. Further, we focus on the conditions under which the implication biharmonic geodesic becomes true. The first case to be investigated is the one of smooth closed curves. In the particular case of Riemannian spaces, this turns out to be the known condition of nonpositive sectional curvature; still, in the general Finslerian case, this condition is more complicated, involving the Cartan and Landsberg tensors. A special attention is paid to two-dimensional Finsler spaces.

3. Conclusion

From the review of the above papers is is seen that the theory of special Finsler spaces was introduced by M. Matsumoto [17] and their properties were studied by C. Shibata [18]. The conformal theory of Finsler spaces was initiated by M.S. Kneblman [20]. The present day authors like [1], [2], [3], etc. have tried to carry out analytical study of special Finsler Spaces.

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